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Exactly Solved Model of Self-Organized Critical Phenomena

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We define a variant of the model of Bak, Tang, and Wiesenfeld of self-organized critical behavior by introducing a preferred direction. We characterize the critical state and, by establishing equivalence to a voter model, determine the critical exponents exactly in arbitrary dimension d. The upper critical dimension for this model is three. In two dimensions the model is equivalent to an earlier solved special case of directed percolation.

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The concept of a self-organized criticality[1-3] has attracted some attention recently as it may explain the ubiquity of power-law correlations in nature, e.g., the occurrence of fractal structures[4] in space and $1/f$ noise[5] in time. Lattice automation models that display this self-organized critical (SOC) behavior have been studied by Bak, Tang, and Wiesenfeld (BTW) using Monte Carlo methods[1] and mean-field theory.[2] Hwa and Kardar (HK) have studied a similar model[6] where they determine the (presumably exact) critical exponents by nonrigorous dynamical renormalization-group arguments. However, no exactly solvable cases are known so far except for the trivial one-dimensional problem.

In this Letter we define a new model of self-organized critical behavior by incorporating a preferred direction. This model has the advantage of being exactly solvable, and we are able to determine the critical exponents and the two-point correlation function exactly in any dimension.

In the BTW model of SOC behavior, to each site $\mathbf{X}$ on a d-dimensional hypercubic lattice is associated an integer variable $z(\mathbf{X})$. The dynamics of the model is discrete, nonlinear, and diffusive, and is defined by the rule that if any $z(\mathbf{X})$ exceeds a critical value, z_c, then the variables at m neighboring sites, denoted {nn}, increase by 1, while $z(\mathbf{X})$ decreases by m: If $z(\mathbf{X}) \geq z_c$ then

$$z(\mathbf{X}) \rightarrow z(\mathbf{X}) - m\,,$$
$$z(\mathbf{X}') \rightarrow z(\mathbf{X}') + 1 \text{ for } \mathbf{X}' \in \{\text{nn}\}\,. \qquad (1)$$

Without loss of generality we choose $z_c = m$, and in the steady state, the z's then take the m values $0, 1, 2, \ldots, m-1$. In d dimensions, we have explicitly[1] the BTW model:

$$m = 2d \text{ and } \{\text{nn}\} = \{\mathbf{X} \pm \mathbf{e}_k,\ k = 1, 2, \ldots, d\}\,, \qquad (2)$$

where $\mathbf{e}_k$ are the basis vectors of the lattice.

Starting from an arbitrary configuration, if particles are added $[z(\mathbf{X}) \rightarrow z(\mathbf{X}) + 1]$ at random sites $\mathbf{X}$, the BTW model eventually reaches a SOC state which is statistically stationary: Subsequent addition of a particle leads to reorganization on all length scales.[1] Such models may be expected to apply to a wide variety of physical phenomena[3,7,8] and, in particular, to the dynamics of "sandpiles" where it has been suggested[1] that the $z(\mathbf{X})$'s may be interpreted as local slopes. However, the existence of a preferred direction (that of the steepest descent, say) is not reflected in the isotropic dynamics of Eq. (1). This is expected to be a *relevant* perturbation since in other related problems such as percolation, the critical behaviors of directed and undirected problems are known to differ.

The simplest way to introduce directionality in the BTW model is to restrict {nn} to sites along the preferred direction. This defines our model A:

$$m=d \text{ and } \{\text{nn}\}=\{\mathbf{X}+\mathbf{e}_k,\ k=1,2,\ldots,d\}. \quad (3)$$

Physically $z(\mathbf{X})$ may be interpreted as the height of a sand column at the point $\mathbf{X}$ above some reference level, the direction of steepest descent being $(1,1,\ldots,1)$. It may be more realistic to assume a finite but smaller probability of sand from $\mathbf{X}$ dropping at backward neighbors as well. Here we study the simpler case when such transitions are completely forbidden. If the $z(\mathbf{X})$'s are interpreted as local height differences, similar rules for the evolution of these variables can be written, but these are more complicated, and do not correspond to the isotropic BTW model either.

We define the longitudinal coordinate of a site $\mathbf{X} \equiv (X_1,X_2,\ldots,X_d)$ as $T(\mathbf{X})=\sum X_i$ and consider model A in a finite lattice confined to the region $0\le T<L$. In transverse directions we assume finite extent with periodic boundary conditions, the number of sites on each constant-T surface being S. Particles are added at random on the "top" (i.e., at the surface $T=0$) and drop off the pile at $T=L$. Some of the terminology from the sandpile analogy is convenient. Thus when $z(\mathbf{X})$ exceeds z_c, that column *topples* and the effect of a toppling is to create an *avalanche,* which is characterized by its *duration t,* which is the number of constant-T surfaces affected, while its *mass, M,* is the total number of sites that toppled.

Any configuration satisfying $0\le z(\mathbf{X})\le d-1$ for all $\mathbf{X}$ is a stable configuration. The total number of such configurations is d^{LS}. We now show that in the SOC state, each of these configurations occurs with equal probability. Let C_0 be the starting configuration, with the ith particle being added at (the arbitrary) site P_i and resulting in the new stable configuration C_i. Then C_i is determined uniquely by the dynamics (1), given P_i and C_{i-1}. The dynamics is also *invertible.* This is easily seen: On the top layer C_{i-1} differs from C_i only at site P_i, with $z(P_i)$ in C_{i-1} being less than its value in C_i by $1(\text{mod} d)$. Other layers in C_{i-1} are the same as in C_i if there was no toppling at P_i; otherwise the z's in the second layer in C_{i-1} are the same as in C_i, except at the d forward neighbors of P_i whose heights are less by $1(\text{mod} d)$ than their values in C_i. And so on for subsequent layers. Thus, given C_i and P_i, we can uniquely determine C_{i-1}.

The invertibility of the deterministic dynamics is crucial for the exact solvability of the model. Thus, given a state C_i, there are precisely S possible choices of C_{i-1} and C_{i+1} corresponding to S possible choices of P_i and P_{i+1}. It then follows from the master equation for the evolution of probabilities of configurations that the state prepared with

$$\mathcal{P}(C_0=a)=\text{const, independent of } a\,, \quad (4)$$

is invariant in time ($\mathcal{P}$ denotes probability). This invariant state is degenerate. The sum of z's along a constant-T surface is conserved modulo d for $T>0$. These additional conservation laws imply that the invariant state (4) can be decomposed into d^{L-1} extremal invariant states; in the thermodynamic limit, differences between extremal states are unimportant, and may be ignored.

Let $G_0(\mathbf{X};\mathbf{Y})$ be the probability that site $\mathbf{X}$ topples over in the SOC state in the avalanche created by adding a particle at site $\mathbf{Y}$. Clearly, the conditional probability that $\mathbf{X}$ will topple, given that r of its backward neighbors have toppled, is r/d. Such stochastic processes, termed voter models,[9] have been studied earlier and are exactly solvable. In our case, $G_0(\mathbf{X};\mathbf{Y})$ satisfies the linear equation

$$G_0(\mathbf{X};\mathbf{Y})=\frac{1}{d}\left[\sum_{i=1}^{d} G_0(\mathbf{X}-\mathbf{e}_i;\mathbf{Y})+\delta_{\mathbf{X},\mathbf{Y}}\right] \quad (5)$$

which is easily solved to give, in the thermodynamic limit,

$$G_0(\mathbf{X};\mathbf{0})=\left[T(\mathbf{X})!\Big/\prod_i X_i!\right]d^{\,-1-T} \quad (6)$$

if all X_i are nonnegative, and zero otherwise. Summing over all sites $\mathbf{X}$ on the surface $T=t$ (the primed summation below), we get

$$\sum{}' G_0(\mathbf{X};\mathbf{0})=1/d \text{ for all } t\ge 0 \quad (7)$$

since in the steady state the average flux in and out of a constant-T surface are equal, and each equals one particle per added site. (One toppling gives a flux of d particles.)

For finite S there is a nonzero probability that all sites on some surface $T=T_0$ will topple. This makes all sites on surface T_0+1 topple as well and this self-perpetuating state leads to infinite avalanches. From Eq. (7) it follows immediately that in the limit of large L and finite S the

$$\mathcal{P}(\text{infinite avalanche})=1/dS\,. \quad (8)$$

The trivial case $d=S=1$ is easily verified.

If S is infinite, the probability that in the SOC state the avalanche has a duration greater than t varies as $t^{-\alpha}$ for large t, where α is some exponent. Let $m(t)$ be the expected number of sites that topple on the surface $T=t$ given that at least one site does. Then $m(t)$ must vary as t^{α} to keep a constant mean flux. Hence the mass M of an avalanche of total duration t should scale as $t^{1+\alpha}$, which implies that the probability that the avalanche exceeds M varies as $M^{2-\tau}$ with $\tau-2=\alpha/(1+\alpha)$.

To determine α, first consider $d=2$. In this case, there is an additional simplification that all sites that topple on a line $T=\text{const}$ must be contiguous (Fig. 1). Consequently, the cluster of toppled sites in an avalanche has no holes; furthermore, the boundary of such a cluster is

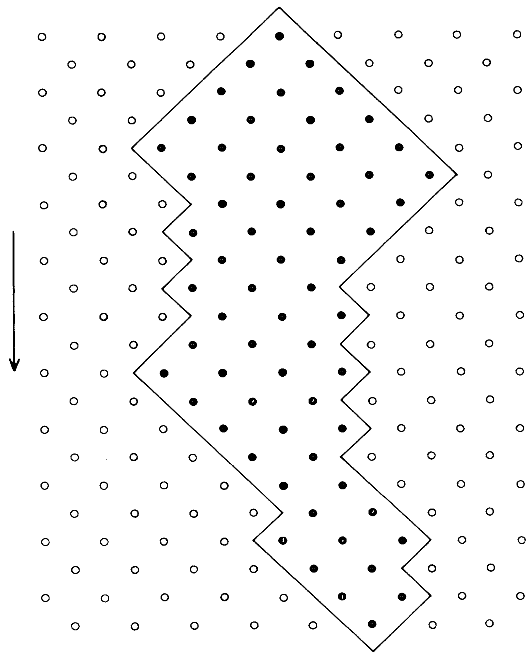

FIG. 1. Cluster of sites in a typical avalanche formed under the evolution dynamics of model A in two dimensions. Solid circles denote sites that have toppled. This cluster has duration 22 and mass 64, and the random walks that make the cluster boundary are shown.

formed by the path of two annihilating random walkers starting at the origin, each taking steps along $\mathbf{e}_1$ and $\mathbf{e}_2$ with equal probability. The model is thus exactly equivalent to a restricted case of directed percolation studied earlier by Domany and Kinzel.[10] For the distribution of durations of avalanches we get

$$\mathcal{P}(t)=(2t)!/[t!(t+1)!]2^{-2t-1}. \tag{9}$$

For large t_0, the probability that $t \geq t_0$ varies as $t_0^{-1/2}$. Hence for $d=2$,

$$\alpha=\tfrac{1}{2}. \tag{10}$$

To calculate α for general d, we define the three-point function $G(\mathbf{X}_1,\mathbf{X}_2;\mathbf{X}_0)$ as the probability that sites $\mathbf{X}_1$ and $\mathbf{X}_2$ both belonging to the same constant-T hypersurface topple after a particle is added at $\mathbf{X}_0$. This function obeys the difference equation

$$G(\mathbf{X}_1,\mathbf{X}_2;\mathbf{X}_0)=d^{-2}\sum\sum G(\mathbf{X}_1-\mathbf{e}_i,\mathbf{X}_2-\mathbf{e}_j;\mathbf{X}_0) \quad \text{for } \mathbf{X}_1\neq\mathbf{X}_2. \tag{11}$$

These equations are automatically satisfied if G is of the form

$$G(\mathbf{X}_1,\mathbf{X}_2;\mathbf{X}_0)=\sum_{\mathbf{Y}} f(\mathbf{Y}-\mathbf{X}_0)G_0(\mathbf{X}_1;\mathbf{Y})G_0(\mathbf{X}_2;\mathbf{Y}), \tag{12}$$

where the unknown functions $f(\mathbf{Y})$ are determined by the conditions

$$G(\mathbf{X},\mathbf{X};\mathbf{X}_0)=G_0(\mathbf{X};\mathbf{X}_0). \tag{13}$$

Summing Eq. (13) over all $\mathbf{X}$ such that $T(\mathbf{X})=T(\mathbf{X}_0)+t$ we get, using Eq. (7),

$$\sum_{t'=0}^{t} F(t')K(t-t')=1/d, \tag{14}$$

where

$$K(t)=\sum{}' G_0(\mathbf{X};\mathbf{0})G_0(\mathbf{X};\mathbf{0})$$

and $F(t)=\sum{}' f(\mathbf{X})$. The primed summations above are over all $\mathbf{X}$ on the surface $T=t$. Equation (14) determines $f(t)$ for all t recursively. Since for large t, $K(t)\approx t^{(1-d)/2}$, it follows that as $t\to\infty$, $F(t)\approx$ const for $d\geq 4$, $F(t)\approx 1/\ln t$ for $d=3$, and $F(t)\approx t^{-1/2}$ for $d=2$.

The mean square flux out of surface $T=t$ is given by

$$\sum_{\mathbf{X}_1,\mathbf{X}_2}{}' G(\mathbf{X}_1,\mathbf{X}_2;\mathbf{X}_0)=\sum_{t'=0}^{t} F(t'). \tag{15}$$

Since this should vary as L^{α}, we get

$$\alpha=1 \text{ for } d\geq 3 \tag{16}$$

with additional logarithmic corrections for $d=3$.

The problem on other directed lattices can be treated similarly. For the triangular lattice in $d=2$ we have

$$m=3 \text{ and } \{\text{nn}\}=\{\mathbf{X}+\mathbf{e}_1,\mathbf{X}+\mathbf{e}_2,\mathbf{X}+\mathbf{e}_1+\mathbf{e}_2\}. \tag{17}$$

Similar to the case of the square lattice, all configurations with $z(\mathbf{X})$ taking values 0, 1, or 2 are stable, and occur with equal probability in the SOC state. It is easy to see that for this model also avalanches leave no holes and thus the critical exponents are exactly the same as for model A.

To explore the question of universality in such models, we have also studied a "partially directed" variant of this problem on a square lattice, defined by (model B)

$$d=2,\ m=3, \text{ and } \{\text{nn}\}=\{\mathbf{X}+\mathbf{e}_1,\mathbf{X}-\mathbf{e}_1,\mathbf{X}+\mathbf{e}_2\}. \tag{18}$$

Particles are added at random along the line $\mathbf{X}_2=0$. For model B, in contrast to the earlier $d=2$ cases studied, the dynamical rules appear to allow holes in avalanche clusters. The SOC state is *not* characterized as simply as for model A. We find that configurations having adjacent 0's or two 0's separated by a string of 1's on a constant-$\mathbf{X}_2$ line are strictly forbidden, but the configurations that *are* permitted occur with *equal* probability. In fact, all configurations that could lead to holes in avalanches are forbidden *in the SOC state.* The exponents are thus the same as in model A for $d=2$. For $d\geq 3$ the situation is more complicated, and is presently being investigated.

Although we, like Hwa and Kardar,[6] modify the BTW model so as to incorporate the directionality of the average flow, note that the critical exponents obtained here differ from those obtained by HK, as does the value of the upper critical dimension—four for HK as opposed to three in the present work. While HK postulate a nonlinear stochastic evolution equation for suitably coarse-grained local variables of the system, we start with a reasonable (albeit specific) explicit discrete evolution rule and derive linear equations. Which model is more suitable in what domain of application is not clear at present.

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